Advanced Statistical Physics

Lecture 2 : Fundamental Concepts - Part II

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Before we begin

- Please, write down your expectations from the course.
- Typos of the course
- Recommendations are always welcome.

Recap

- Motivation for studying of statistical physics
- Review on thermodynamics
- A summary on thermodynamics through statistical relations

A crash course on statistical mechanics.

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A crash course on statistical mechanics.

"With thermodynamics, one can calculate almost everything crudely; with kinetic theory, one can calculate fewer things, but more accurately; and with statistical mechanics, one can calculate almost nothing exactly." Eugene Paul Wigner

The heroes

The micro-canonical ensemble

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An ensemble that follows the probability distribution (6) to describe isolated systems, is called *the microcanonical ensemble*.

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Question : What is the probability P_r of finding system A in any one particular microstate r of energy E_r ?

Answer :

$$P_r = \frac{e^{-\beta E_r}}{\sum_r e^{-\beta E_r}}$$

This expression is central in statistical physics, the exponential factor $e^{-\beta E_r}$ is called the **“Boltzmann factor”** and the probability distribution is the **“canonical distribution”**. Any ensemble describing systems interacting with a heat bath characterized with a temperature T, following the canonical distribution, is called the **“canonical ensemble”**.

Application : Paramagnetism

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What is the mean magnetic momentum $\bar{\mu}_H$ of such an atom ?

Application : Paramagnetism

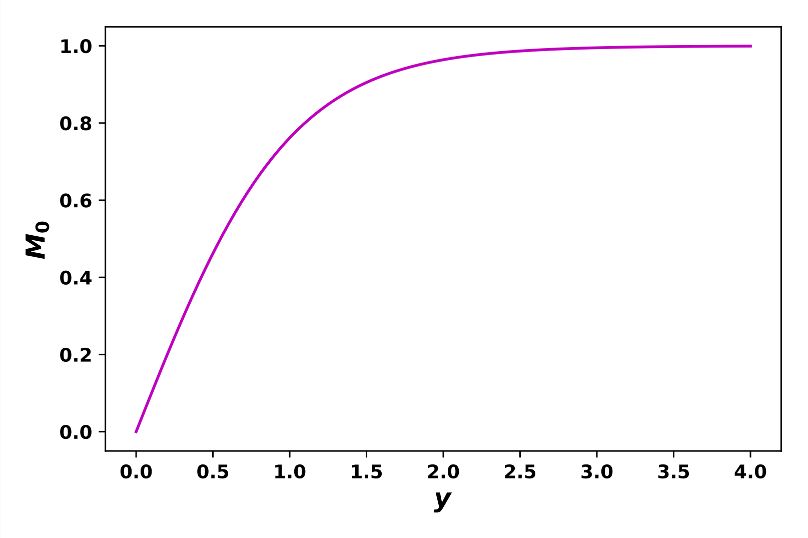

Figure 1: Magnetization with respect to y.

Other ensembles

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A particular example is **the grand canonical ensemble**, in which $X = N$, meaning that the system can exchange the energy E and the particles N. The parameter β is the temperature of the ensemble, and $\alpha = \frac{-\mu}{kT}$, where μ is **the chemical potential** of the reservoir.

Our lord and savior “Z”

Connections with thermodynamics

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$$Z = \sum_r e^{-\beta E_r}. \tag{1}$$

"Z" is a sum over states, called **the partition function**. Z is used as symbol for the parition function due to its name in German "*Zustandssumme*"

Application : The one dimensional harmonic oscillator

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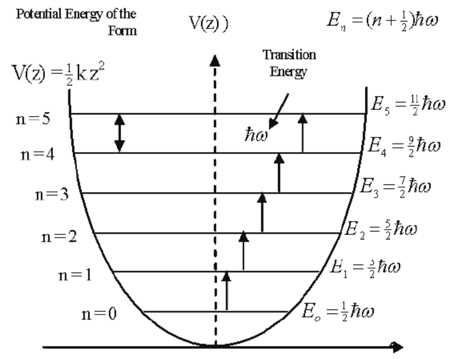

Figure 2: Energy levels of the 1D harmonic oscillator

Application : The one dimensional harmonic oscillator

$$Z = \frac{1}{2\sinh(-\beta\frac{\hbar\omega}{2})}$$

$$\bar{E} = -\frac{\partial \ln Z}{\partial \beta} = -\frac{\hbar\omega}{2}\coth(-\beta\frac{\hbar\omega}{2}),$$

$$S = k\left(\ln Z + \beta\bar{E}\right) = k\left[-\ln\left(2\sinh\left(\frac{\hbar\omega}{2kT}\right)\right) + \frac{\hbar\omega}{2kT}\coth\left(\frac{\hbar\omega}{2kT}\right)\right],$$

$$F = -kT\ln Z = kT\ln\left(2\sinh\left(\frac{\hbar\omega}{2kT}\right)\right)$$

Statistics of Ideal Gases

Statistical distribution functions

Consider a gas of identical particles characterized by a volume V, temperature T and a number of possible quantum states R. Each particle can be in quantum state labeled by r with an energy ϵ_r and the number of particles occupying the state r is n_r.

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The mean number of particles in a state s is :

$$\bar{n}_s = -\frac{1}{\beta}\frac{\partial \ln Z}{\partial \epsilon_s}$$

Maxwell-Boltzmann statistics

The Maxwell-Boltzmann statistics takes into account the distinguishability of particles, as a result the partition function given by :

$$Z = \sum_{n_1, n_2, \ldots} \frac{N!}{n_1! n_2! \ldots} e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \ldots)},$$

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The mean number of particles in a state s is :

$$\bar{n}_s = N \frac{e^{-\beta \epsilon_s}}{\sum_r e^{-\beta \epsilon_r}}.$$

This distribution is called : the **Maxwell-Boltzmann distribution**.

Photon statistics

Now, we will consider a *special* case of the Bose-Einstein statistics where the total number N is not fixed. The partition function is given by :

$$Z = \sum_R e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \dots)},$$

then :

$$\ln Z = -\sum_r \ln\left(1 - e^{-\beta\epsilon_r}\right).$$

The mean number of particles in the state s :

$$\bar{n}_s = \frac{1}{e^{\beta\epsilon_s} - 1}.$$

This is called the **“Planck distribution”**.

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$$= \frac{\sum_{n_s} n_s e^{-\beta n_s \epsilon_s} \sum^{(s)}_{n_1\dots n_2} e^{-\beta(n_1\epsilon_1+n_2\epsilon_2+\dots)}}{\sum_{n_s} e^{-\beta n_s \epsilon_s} \sum^{(s)}_{n_1\dots n_2} e^{-\beta(n_1\epsilon_1+n_2\epsilon_2+\dots)}}. \tag{3}$$

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$$\bar{n}_s = \frac{1}{e^{\alpha+\beta\epsilon_s}+1}. \tag{4}$$

This is called the **Fermi-Dirac distribution**.

Bose-Einstein statistics

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$\alpha = -\beta\mu$ is again the chemical potential and the formula (**??**) represents the **“Bose-Einstein statistics”**.

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- For BE statistics :
  It is fine to have multiple particles in the same state, so to reach the lowest energy of the whole gas (at $T \to 0$), all the particles need to be placed in their lowest-lying state of energy ϵ_1.
- For FD statistics :
  One particle per state. Then, in order to reach the lowest energy state of the whole gas, one need to populate all the single-particle states starting from the state of the lowest energy ϵ_1 until all the particles are accommodated. The gas as a whole is in its state of lowest energy, but there are particles that have a very high energy compared to ϵ_1.

Remark : The classical limit of quantum statistics

As $\beta \to 0$ in FD and BE statistics, large energies ϵ_r contribute to the sum. To prevent this sum from exceeding N, α must become large enough to that each ϵ_r is sufficiently small. That is $e^{\alpha+\beta\epsilon_r} \gg 1$, then FD and BE statistics reduces to :

$$\bar{n}_r = N\frac{e^{-\beta\epsilon_r}}{\sum_r e^{-\beta\epsilon_r}}.$$

Then, in the classical limit BE and FD statistics reduces to MB statistics.

Further readings

- Chapter 6, 7 and 9 of “F. Reif - Fundamentals of Statistical and Thermal Physics.”
- Chapter 5, 6, 7 and 9 of “C. Ngo H. Ngo - Physique statistique”.

Next lecture is dedicated to exercises.

Thank you for your attendance and your attention.