Advanced Statistical Physics

Part II : Phase Transitions and Critical Phenomena

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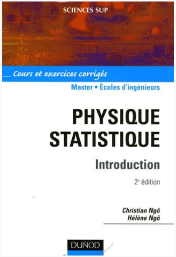
1. C.Ngo, H.Ngo - Physique Statistique - Chapter 10 & 11.
2. Nigel Goldenfeld - Lectures On Phase Transitions And The Renormalization Group - Chapter 2, 3 & 5.



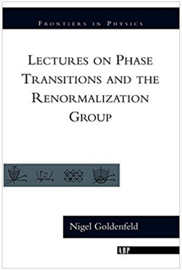


Table of contents

1. Introduction
2. How Phase Transitions Occur in Principle
3. How Phase Transitions Occur in Practice
4. Landau Theory of Phase Transitions.

Introduction

What is a Phase ? What is a phase transition ?

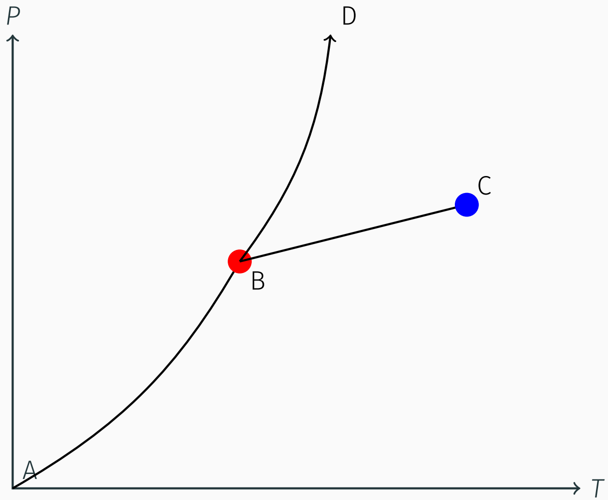


Figure 1: Phase diagram of water.

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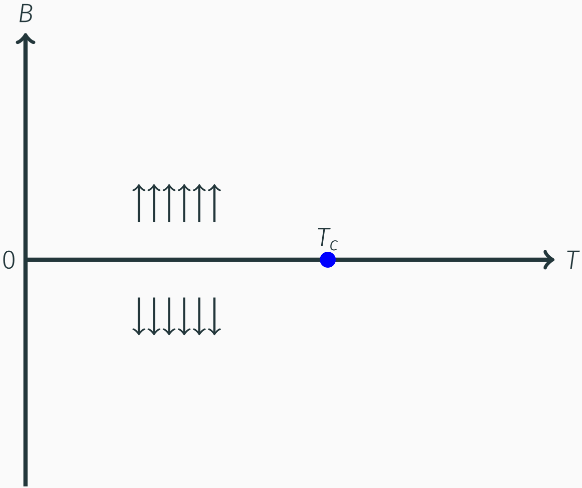

Figure 2: Phase diagram of a ferromagnetic material.

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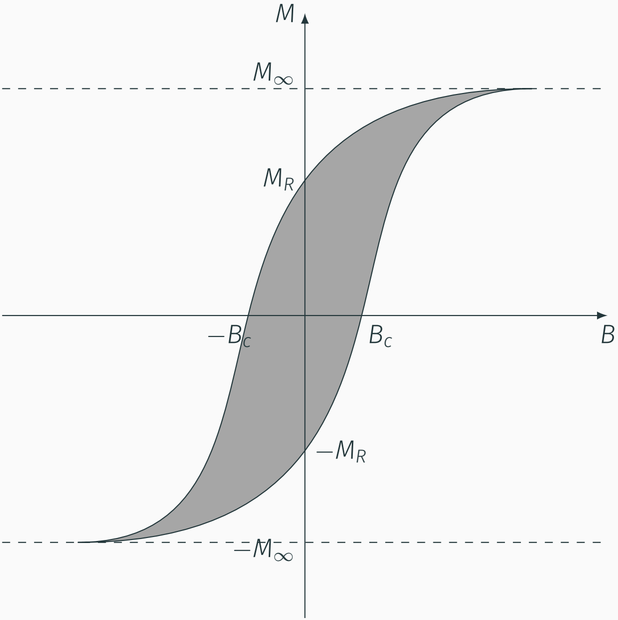

How Phase Transitions Occur in Principle

$f(x)$ is a concave function of x if :

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2} \qquad \text{for all } x_1 \text{ and } x_2.$$

Which means that the chord joining the points $f(x_1)$ and $f(x_2)$ lies above of $f(x)$ for all x in $x_1 < x < x_2$.

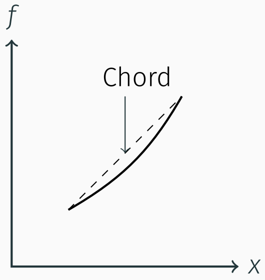

Figure 3: Convex function.

Preliminaries : Concavity

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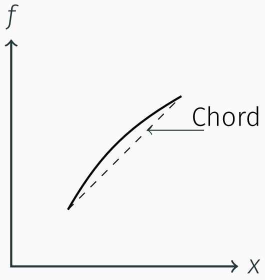

Figure 4: Concave function

Consequences of the convexity and the concavity

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Consequences of the convexity and the concavity

The specific heat and the susceptibility (for magnetic materials) are positive thermodynamical response function, which implies that the free energy F is convex. This is a direct consequence of Le Chatelier's principle for stable equilibrium which states that : if a system is in thermal equilibrium any small spontaneous fluctuation in the system parameter, the system gives rise to certain processes that tends to restore the system back to equilibrium.

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Preliminaries : Thermodynamic limit

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When Ω is finite, there is no information about phase transitions or phases as this phenomena occurs *theoretically* in the thermodynamic limit that is $\Omega \to \infty$.
The existence of the thermodynamic limit is not trivial as it fails to exist for some systems.

Preliminaries : Existence of the Thermodynamic limit

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$$\begin{aligned} E &= \int_0^R \left(\frac{4}{3}\pi r^3 \rho\right) \frac{A}{r} 4\pi r^2 \rho dr \\ &= A\frac{(4\pi)^2}{15}\rho^2 R^5. \end{aligned}$$

The energy per unit volume is then : $\epsilon = A\frac{(4\pi)^2}{15}\rho^2 R^2$, which diverges as $R \to \infty$.

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Inverse square law forces like gravity and electrostatics are too long-ranged to permit thermodynamic behaviour

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Taking the limit $R \to \infty$, the system is stable only when $m > d$. The thermodynamic limit exist then only when $m > d$.

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Then, a phase is just a region of analycity of $f_b[K]$ and loci of co-dimension $C = D - D_s = 1$ is called a phase boundary.

Preliminaries : Phase boundaries and Phase transitions

$f_b[K]$ can be used also to classify phase transitions :

- **First order phase transitions**
  If one or more $\partial f_b/\partial K_i$ are discountinous across a phase boundary, the transition is first order.
- **Continous phase transitions**
  If the first derivative of $f_b[K]$ is countinous across the phase boundary, the transition is said to be countinous phase transitions (or a second order phase transition)

The role model

The Ising model

The Ising model can be written as :

$$-H_\Omega = \sum_{i \in \Omega} H_i S_i + \sum_{ij} J_{ij} S_i S_j \sum_{ijk} K_{ijk} S_i S_j S_k + \dots \tag{1}$$

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The free energy is given by : $F_\Omega(T, H_i, J_{ij}) = -k_b T \log \mathrm{Tr}\, e^{-\beta H_\Omega}$.
The thermodynamical properties can be calculated through F_Ω, for example the magnetization at site i is :

$$\frac{\partial F_\Omega}{\partial H_i} = -k_B T \frac{1}{\mathrm{Tr}\, e^{-\beta H_\Omega}} \mathrm{Tr}\, \frac{S_i}{k_b T} e^{-\beta H_\Omega} = -\langle S_i \rangle_\Omega$$

The Ising model : Spin-reversal symmetry

The Ising model is Z_2 symmetric. That is a rotation of π of all the spins, leave the system energy unchanged. Mathematically, this implies that :

$$H_\Omega(H, J, S_i) = H_\Omega(-H, J, -S_i).$$

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Thus :

$$\begin{aligned}
Z_\Omega(-H, J, T) &= \sum_{S_i=\pm 1} e^{-\beta H_\Omega(-H,J,S_i)} \\
&= \sum_{S_i=\pm 1} e^{-\beta H_\Omega(-H,J,-S_i)} \\
&= \sum_{S_i=\pm 1} e^{-\beta H_\Omega(H,J,S_i)} \\
&= Z_\Omega(H, J, T).
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The free energy is then :

$$F(H, J, T) = F(-H, J, T)$$

The Ising model : Sub-lattice symmetry

This symmetry emerges at zero magnetic field ($H = 0$). We divide the lattice into two interpenetrating lattices A and B. The spins of lattice A interacts only with the ones in the lattice B

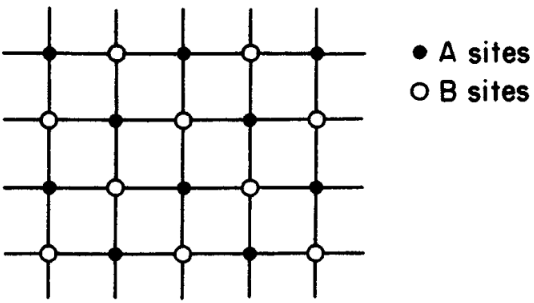

Figure 5: Two interpenetrating lattices A and B

The Ising model : Sub-lattice symmetry

The Hamiltonian is $H_\Omega(0, J, S_i^A, S_i^B) = -J \sum_{\langle ij \rangle} S_i^A S_j^B$.
The sub-lattice symmetry implies that :

$$H_\Omega(0, -J, S_i^A, S_i^B) = H_\Omega(0, J, -S_i^A, S_i^B) = H_\Omega(0, J, S_i^A, -S_i^B).$$

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In zero field we write the partition function :

$$\begin{aligned} Z_\Omega(0, -J, T) &= \mathrm{Tr}\, e^{-\beta H_\Omega(0, -J, T)} \\ &= \sum_{S_i^A} \sum_{S_j^B} e^{-\beta H_\Omega(0, -J, S_i^A, S_i^B)} \\ &= \sum_{S_i^A} \sum_{S_j^B} e^{-\beta H_\Omega(0, J, -S_i^A, S_i^B)} \\ &= \sum_{S_i^A} \sum_{S_j^B} e^{-\beta H_\Omega(0, J, S_i^A, S_i^B)} \\ &= Z_\Omega(0, J, T). \end{aligned}$$

Thus, the free energy satisfies the following :

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The sub-lattice symmetry implies that the thermodynamics of the ferromagnetic Ising model and that of the anti-ferromagnetic Ising model are the same at zero magnetic field.

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The Ising model : 0T Phase diagram

Consider the Ising model in d-dimensions at $T = 0$, then F=E.

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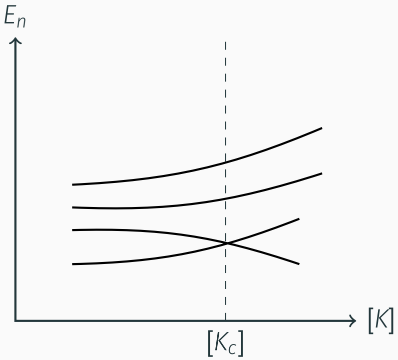

Figure 6: The mechanism of level crossing

The Ising model : 0T Phase diagram

The ground state is obtained for $J > 0$ by noticing that $-J\sum_{\langle ij \rangle} S_i S_j$ is minimized when $S_i = S_j$ and the term $-H\sum_i S_i$ is minimized by $S_i = +1$ when $H > 0$ and $S_i = -1$ when $H < 0$.

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$$S_i = \begin{cases} +1 & H > 0, J > 0; \\ -1 & H < 0, J > 0. \end{cases}$$

The magnetization is then :

$$M_\Omega = \frac{1}{N(\Omega)} \sum_{i\in\Omega} S_i = \begin{cases} +1 & H > 0; \\ -1 & H < 0. \end{cases}$$

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A phase transition occurs at zero temperature and at zero magnetic field.

The Ising model : 1D phase diagram

Consider N spins pointing all up.

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What effect does this have on the thermodynamics ? $\Delta E = 2J$, while the domain wall introduced can be placed in N different positions, the entropy is then $\Delta S = k_b \ln N$. Therefore, the change in the free energy is :

$$\Delta F = \Delta E - T\Delta S = 2J - k_b T \ln N.$$

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There is no long range order. Thus, the 1D Ising model has no phase transitions at $H = 0$

The Ising model : 2D phase diagram

We consider again a domain of flipped spins, in a background of spins with long range order, but this time the domain in two dimensional. What is the energy difference ? and what is the entropy?

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We consider again a domain of flipped spins, in a background of spins with long range order, but this time the domain in two dimensional. What is the energy difference ? and what is the entropy?

The internal energy change of a domain of length L is $\Delta E = 2JL$. The entropy can be estimated by enumerating the different possibilities of the domain wall, it turns out that this number is proportional to the coordinate number of the lattice z. The entropy is then $\Delta S = k_b L \log(z-1)$ and the free energy is:

$$\Delta F = 2JL - (\log(z-1))\, k_b T L.$$

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We can speak of phase transitions at finite temperatures only in the 2D Ising model or above

The impossibility theorem

The impossibility of phase transitions can be seen immediately from the spin-reversal symmetry of the Ising model. We know that the free energy satisfies the following :

$$F_\Omega(H, J, T) = F_\Omega(-H, J, T),$$

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and the magnetization satisfies :

$$N(\Omega)M_\Omega(H) = -\frac{\partial F_\Omega(H)}{\partial H} = -\frac{\partial F_\Omega(-H)}{\partial H} = \frac{\partial F_\Omega(-H)}{\partial - H} = -N(\Omega)M_\Omega(-H).$$

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This is the **impossibility theorem** it shows that the magnetization in $H = 0$ should be zero, a result that contradicts our previous finding. What has gone wrong ?

When $N \to \infty$ the free energy develops a discontinuity in its first derivative, and knowing the fact that $F(H)$ is a convex function, the condition $F(H) = F(-H)$ does not *imply* $M(0) = 0$, for that to happen we need to add the assumption of the smoothness of the free energy at $H = 0$ and that the left and right derivatives are equal.

Spontaneous Symmetry Breaking

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$$F(H) = F(0) + O(H^p) \qquad\qquad p > 1$$

and

$$\lim_{\epsilon \to 0} \frac{F(+\epsilon) - F(0)}{\epsilon} = \lim_{\epsilon \to 0} \frac{F(-\epsilon) - F(0)}{\epsilon} = 0$$

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We can then turn around the impossibility theorem and still satisfy the analytical properties of the free energy by writing :

$$F(H) = F(0) - M_s|H| + O(H^p) \qquad\qquad p > 1$$

which is not differentiable at $H = 0$, but still hold the convexity property as depicted Fig. 7

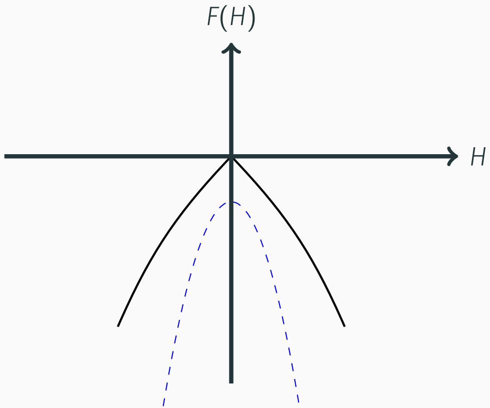

Figure 7: The free energy as function of H for a finite system (dashed line) and for an infinite system (solid line)

Spontaneous Symmetry Breaking

$$\frac{\partial F}{\partial H} = \begin{cases} -M_s + O(H^{p-1}), & H > 0 \\ -M_s + O(H^{p-1}), & H < 0. \end{cases}$$

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Notice :

$$\lim_{N(\Omega)\to\infty} \lim_{H\to 0} \frac{1}{N(\Omega)} \frac{\partial F_\Omega(H)}{\partial H} = 0 \quad \text{and} \quad \lim_{H\to 0} \lim_{N(\Omega)\to\infty} \frac{1}{N(\Omega)} \frac{\partial F_\Omega(H)}{\partial H} \neq 0$$

are not equal. Even tough the Hamiltonian is invariant under spin reversal, the expectation values do not follow this symmetry, so that $\langle S_i \rangle \neq 0$ and : $M = \lim_{N\to\infty} \frac{1}{N(\Omega)} \sum_i \langle S_i \rangle \neq 0$.

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How Phase Transitions Occur in Practice

Transfer Matrix

We start with the 1D nearest-neighbors Ising model.

$$-H_\Omega = H \sum_{i \in \Omega} S_i + \sum_{\langle ij \rangle} J_{ij} S_i S_j, \qquad J > 0.$$

Let $h = \beta H$ and $K = \beta J$, and suppose periodic boundary conditions, that is $S_{N+1} = S_1$.

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Let $h = \beta H$ and $K = \beta J$, and suppose periodic boundary conditions, that is $S_{N+1} = S_1$. Then, the partition function is :

$$\begin{aligned} Z_N(h,K) &= \operatorname{Tr}\exp\left[h\sum_i S_i + K\sum_i S_iS_{i+1}\right] \\ &= \sum_{S_1}\cdots\sum_{S_N}\left[e^{\frac{h}{2}(S_1+S_2)+KS_1S_2}\right].\left[e^{\frac{h}{2}(S_2+S_3)+KS_2S_3}\right]\ldots\left[e^{\frac{h}{2}(S_N+S_1)+KS_NS_1}\right]. \end{aligned}$$

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Each term in the partition function can be written as a matrix T :

$$T_{S_1 S_2} = e^{\frac{h}{2}(S_1+S_2)+K S_1 S_2},$$

whose elements are :

$$T = \begin{pmatrix} T_{1,1} & T_{1,-1} \\ T_{-1,1} & T_{-1,-1}. \end{pmatrix} = \begin{pmatrix} e^{h+K} & e^{-K} \\ e^{-K} & e^{-h+K}, \end{pmatrix} \qquad (3)$$

Transfer Matrix

the partition function can be re-written in terms of the matrix T as :

$$Z_N(h, K) = \sum_{S_1} \cdots \sum_{S_N} T_{S_1 S_2} T_{S_2 S_3} \ldots T_{S_N S_1}.$$

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Thus :

$$Z_N(h,K) = \mathrm{Tr}\left(T^N\right),$$

since T is real and symmetric, we diagonalize it by writing :

$$T' = S^{-1} T S,$$

where S is a matrix whose rows and columns are eigenvectors of T. Then :

$$T' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where λ_1 and λ_2 are the eigenvalues of T. The cyclic property of the trace operation implies that $\mathrm{Tr}(T') = \mathrm{Tr}(T)$, so that :

$$\mathrm{Tr}\left(T^N\right) = \lambda_1^N + \lambda_2^N.$$

Transfer Matrix

Assuming that $\lambda_1 > \lambda_2$, we have :

$$Z_N(h, K) = \lambda_1^N \left(1 + \left[\frac{\lambda_2}{\lambda_1} \right]^N \right),$$

and taking the thermodynamic limit $N \to \infty$, we get :

$$Z_N(h, K) \approx \lambda_1^N \left(1 + O(e^{-\alpha N}) \right),$$

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Solving Eq. 3, we obtain : $\lambda_{1,2} = e^K \left[\cosh h \pm \sqrt{\sinh^2 h + e^{-4K}} \right]$.

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The free energy of the one dimensional Ising model in an external magnetic field is :

$$\frac{F_N(h,K,T)}{N} = -J - k_B T \log\left[\cosh h + \sqrt{\sinh^2 + e^{-4K}}\right] \qquad (4)$$

Perron's Theorem

Theorem

For an $N \times N$ matrix ($N < \infty$) A, with positive entries A_{ij} for all (i, j), the largest eigenvalue satisfies the following :

1. *real and positive*
2. *non-degenerate*
3. *an analytic function of A_{ij}*

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Inspecting Eq. 4 leads to the conclusion that to have non-zero temperature phase transitions, λ_1 should be either non-analytic, degenerate ($\lambda_1 = \lambda_2$), or $\lambda_1 = 0$. On the other side, the transfer matrix for 1D systems with sufficiently short-ranged interactions satisfy the Perron's theorem, that is $\lambda_1 \neq 0$, $\lambda_1 \neq \lambda_2$ and λ_1 in analytic.
Thus, we immediately conclude that there are no finite temperature phase transitions in the 1D Ising model.

Transfer Matrix : 0T Ising model

At $T = 0\ (K \to \infty):\ \lambda_1 = e^K \left[\cosh h + \sqrt{\sinh^2 h}\left(1 + O(e^{-4K})\right)\right] = e^{K+|h|}$.
Then, the free energy and the magnetization are given by :

$$F = -Nk_BT\,(K + |h|) + O(T^2) = -N\,(J + |H|)\,, \quad M = -\frac{1}{N}\frac{\partial F}{\partial H} = \begin{cases} 1 & H > 0; \\ -1 & H < 0, \end{cases}$$

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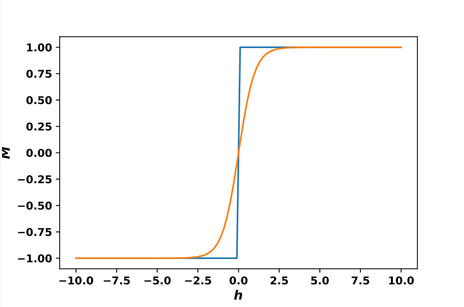

Figure 8: Magnetization vs the magnetic field. Blue line correspond to $T = 0$, while the orange line is for a non-zero temperature.

Thermodynamics

We switch off the magnetic field to calculate the specific heat C_V and the magnetic susceptibility χ_T.

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$$F/N = \begin{cases} -J & T \to 0\,(K \to \infty)\,; \\ -k_B T \log 2 & T \to \infty\,(K \to 0)\,. \end{cases}$$

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$$C_V = \frac{\partial E}{\partial T} = -\frac{1}{k_B T^2}\frac{\partial E}{\partial \beta} = \frac{1}{k_B T^2}\frac{\partial^2 Z}{\partial \beta^2} = \frac{NJ^2}{k_B T^2}\,\mathrm{sech}^2\left(\frac{J}{k_B T}\right).$$

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The heat capacity does not show any singularity, however it exhibit a peak at $J \sim k_B T$, a phenomena known as **Schottky anomaly**

Thermodynamics

The magnetic susceptibility is : $\chi = \frac{\partial M}{\partial H} = \beta \frac{\partial M}{\partial h} = \beta \frac{\partial}{\partial h} \left(\frac{\sinh h}{\sqrt{\sinh^2 h + e^{-4K}}} \right)$

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for small field ($h \to 0$), it reduces to:

$$\chi \sim \begin{cases} \frac{1}{k_b T}, & T \to \infty (\text{Curie's law}); \\ \frac{e^{\frac{2J}{k_B T}}}{k_b T}, & T \to 0 \end{cases}$$

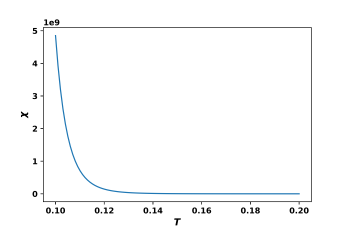

Figure 9: Magnetic susceptibility vs temperature at zero magnetic field.

Correlation functions

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The two point correlation function is defined as :

$$G(i,j) = \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle = \langle S_i S_j \rangle, \quad (\text{For } T > 0 \text{ and } h = 0 : \langle S_i \rangle = \langle S_j \rangle = 0).$$

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A spin at site i is only sensitive to its first neighbor, then :

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Then :

$$\langle S_i S_{i+1} \rangle = \frac{1}{Z} \sum_{\{S\}} S_i S_{i+1} e^{K S_i S_{i+1}} = \frac{\partial \log Z}{\partial K}.$$

Correlation functions

The partition function $Z_{i,i+1}$ is :

$$Z = \sum_{S_i=\pm 1} e^{KS_iS_{i+1}} = 2\left(e^K + e^{-K}\right) = 2^2 \cosh(K).$$

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$$\langle S_i S_{i+1} \rangle = \tanh(K) = \tanh(\beta J).$$

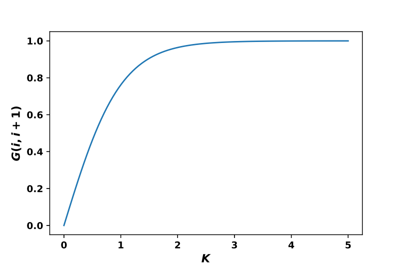

Figure 10: Nearest neighbours correlation function vs temperature. For high temperatures ($K \to 0$) the two spins are less correlated, while for low temperatures ($K \to \infty$) the spins are highly correlated

Correlation functions

It is straightforward to generalize the result beyond $(i, i+1)$:

$$
\begin{aligned}
G(i, i+j) &= \langle S_i S_{i+j} \rangle \\
&= \tanh(K_i) \tanh(K_{i+1}) \ldots \tanh(K_{i+j-1}) \\
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where $\xi = \dfrac{1}{\log(\coth K)}$ is called : **the correlation length**

Correlation functions

The correlation length measures the length over which the spins are correlated. As we approach the transition temperature of the 1D Ising model ($T \to 0$), the correlation length diverges to infinity (ξ diverges exponentially fast), while it approaches zero at high temperatures.

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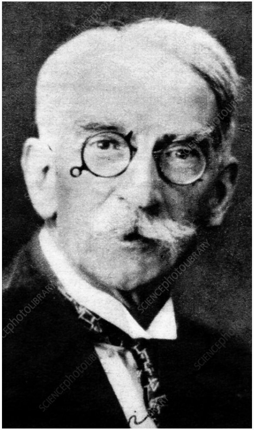
A clear indication of a phase transition is a divergence in the correlation length, for that to happen we need $\lambda_1 = \lambda_2$ (the largest eigenvalue need to be degenerate). This is a general result.

Weiss' Mean Field Theory

Mean field theory :

1. Goal ? : treating interacting statistical mechanical systems
2. Idea ? : for a system of N particles, we replace the interaction between the particles by a mean potential and we forget about fluctuations.
3. Effectiveness ? : when the fluctuation are weak, which not the case around the critical region



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$$Z_\Omega(0, H) = \left[2\cosh\left(\frac{H}{k_BT}\right)\right]^N,$$

the magnetization is :

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Take $J = 0$, a paramagnet. The partition function for such a system is :

$$Z_\Omega(0, H) = \left[2\cosh\left(\frac{H}{k_B T}\right)\right]^N,$$

the magnetization is :

$$M = -\frac{1}{N}\frac{\partial F}{\partial H} = \tanh\left(\frac{H}{k_B T}\right).$$

In this approach, the Ising model can be written as : $H_\Omega = -\sum_i S_i H_i$, where

$$H_i = \underbrace{H}_{\text{magnetic field}} + \underbrace{\sum_j J_{ij}\langle S_j\rangle}_{\text{the mean field}} + \underbrace{\sum_j J_{ij}\left(S_j - \langle S_j\rangle\right)}_{\text{the fluctuations}}.$$

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for a d-dimensional hypercubic lattice we have : $H_i = H + 2dJm$,

Weiss' Mean Field Theory: Critical exponents

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$$M = \tanh(\frac{H}{k_B T} + m\tau) = \frac{\tanh(\frac{H}{k_B T}) + \tanh m\tau}{1 + \tanh(\frac{H}{k_B T}) \tanh m\tau},$$

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for weak H and small m, we can expand Eq. 6 as :

$$\frac{H}{k_B T} \approx M(1 - \tau) + M^3 \left(\tau - \tau^2 + \frac{\tau^3}{3} + \dots\right) + \dots \tag{7}$$

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For zero magnetic field and when $T \to T_c^-$, we have :

$$M^2 \approx 3\frac{T_c - T}{T} + \dots$$

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$$H \sim M^{\delta}.$$

Setting $\tau = 1$ in Eq. 7, we find $\delta = 3$. That is :

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The isothermal magnetic susceptibility also diverges near T_c :

$$\chi_T = \frac{\partial M}{\partial H},$$

from Eq. 7, we get :

$$\frac{1}{k_B T} = \chi_T(1-\tau) + 3M^2\chi_T(\tau - \tau^2 + \frac{1}{3}\tau^3).$$

Weiss' Mean Field Theory: Critical exponents

$M = 0$ for $T > T_c$, then :

$$\chi_T = \frac{1}{k_B}\frac{1}{T - T_c} + \dots \tag{8}$$

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$$M = \sqrt{3} \left(\frac{T_c - T}{T} \right)^{1/2} + \dots$$

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The critical exponent α of the specific heat is calculated from the free energy written in the MFA approximation as :

$$F_m = -k_B T \ln \left[2 \cosh \left(\beta J 2 d m \right) \right],$$

Weiss' Mean Field Theory : Critical exponents

Note that $\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$ and that $M = 0$ for $T > T_c$ while $M = \left(3\frac{T_c - T}{T}\right)^{1/2}$ for $T < T_c$.

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In summary, we have derived the following critical exponents :
$\beta = 1/2$, $\delta = 3$, $\gamma = 1$ and $\alpha = 0$.

Now, we derive an important relationship between the isothermal magnetic susceptibility and the correlations functions.

$$Z = \operatorname{Tr} \exp \left[H\beta \sum_i S_i + \beta J \sum_{\langle ij \rangle} S_i S_j \right] ,$$

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On the other side :

$$\begin{aligned}\chi_T &= \frac{\partial M}{\partial H} = \frac{1}{\beta N}\frac{\partial^2 \log Z}{\partial H^2} = \frac{1}{N}k_B T\left[\frac{1}{Z}\frac{\partial^2 Z}{\partial H^2} - \frac{1}{Z^2}\left(\frac{\partial Z}{\partial H}\right)\right]\\ &= \frac{1}{N}(k_B T)^{-1}\left[\sum_{ij}\langle S_i S_j\rangle - \left(\sum_i \langle S_i\rangle\right)^2\right] = \frac{1}{N}(k_B T)^{-1}\sum_{ij} G(r_i - r_j)\\ &= (k_B T)^{-1}\sum_i G(x_i) = (a^d k_B T)^{-1}\int_\Omega d^d r G(r).\end{aligned}$$

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with $z = r/\xi$. Thus :

$$\xi \sim \left(\frac{T_c - T}{T}\right)^{-\nu}, \tag{10}$$

with $\nu = 1/2$.

The last critical exponent we mention is η, which describe how the point correlation function behave at long distances at the critical point. $G(r)$ for long distances near the critical point is given by : $G(r) \sim r^{-(d-2+\eta)}$, with $\eta = 0$. In principal η can be non zero.

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Exponent	Mean Field	Experiment	2D Ising	3D Ising
α	0	0.110-0.116	0	0.110
β	1/2	0.316-0.327	1/8	0.325 ± 0.0015
γ	1	1.23-1.25	7/4	1.2405 ± 0.0015
δ	3	4.6-4.9	15	4.82
ν	1/2	0.625 ± 0.010	1	0.630
η	0	0.016 - 0.06	1/4	0.032 ± 0.003

Table 1: Critical exponents for the Ising universality class

Can we trust MFT ?

From Tab. 1, there is a clear discrepancy between the critical exponents obtained by the mean field approximation and the experimental result while the exponents of the 3D Ising model are in accordance with experience.

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The mean field approximation is clearly not a good choice for magnetic systems.

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The critical exponents satisfy the scaling relations obtained by thermodynamic considerations, they are given by :

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The precision of the mean field approximation increases as we increase the dimension of the system. In fact, from the scaling relation $2 - \alpha = d\nu$, where d is the dimension of the system, we can deduce the critical dimension at which we get precise results from the mean field approximation. Since $\alpha = 0$ and $\nu = 1/2$, d_c must be 4.

Landau Theory of Phase Transitions.

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1. A theory for all phase transitions.
2. Idea ? : Guessing the potential.
3. Procedure ? : Writing the potential as function of the order parameter m. The minimas with respect to m should describe the thermodynamic properties of the system at the critical point.

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The order parameter m is a quantity used to describe phase transitions, it is zero (non-zero) in the disordered (ordered) phase.

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This theory consists of writing a function L called *Landau free energy* or *Landau functional* in terms of the order parameter η and the coupling constants $\{K_i\}$, where we keep only the terms compatible with the symmetries of the system.

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3. η=0 in the disordered phase, while it is small and non zero in the ordered phase near T_c. Thus, for $T > T_c$ we solve the minimum equation for L by $\eta = 0$ and for $T < T_c$ $\eta \neq 0$ solves the minimum equation. For a homogeneous system we can write :

$$L = \sum_{n=0}^{4} a_n([K])\eta^n. \tag{11}$$

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$a_0([K], T)$ represents the value L in the disordered phase ($\eta = 0$ for $T > T_c$), it describes the degrees of freedom of the system that cannot be understood via the order parameter

Landau Theory

For a_2 and a_4 expanding in temperature near T_c, we obtain :

$$a_4 = a_4^0 + (T - T_c)\, a_4^1 + \ldots$$
$$a_2 = a_2^0 + \frac{(T - T_c)}{T_c} a_2^1 + O\left((T - T_c)^2\right),$$

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Since $\partial^2 L / \partial \eta^2 = 1/\chi = 0$ as $T \to T_c$, one has $a_2^0 = 0$ and :

$$a_2 = \frac{(T - T_c)}{T_c} a_2^1 + O\left((T - T_c)^2\right).$$

The extension to the case $H \neq 0$ for the Ising ferromagnet is immediate

$$L = a\left(\frac{T - T_c}{T_c}\right)\eta^2 + \frac{1}{2} b \eta^4 - H\eta. \tag{14}$$

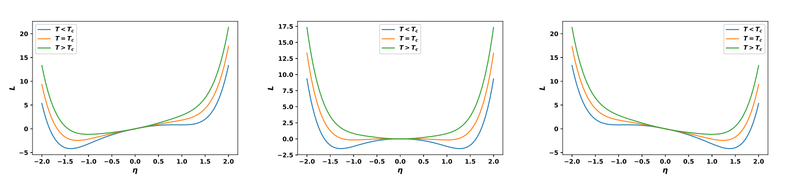
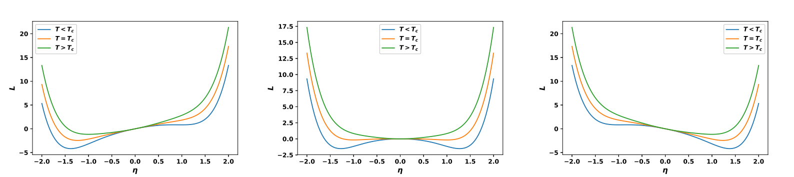


Figure 11: Landau free energy for different values of T and H. From left to right : $H < 0$, $H = 0$ and $H > 0$.



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When $H = 0$ and for $T > T_c$, L has a minimum at $\eta = 0$. When $T = T_c$ Landau potential has zero curvature at $\eta = 0$ while $\eta = 0$ is still the global minimum. For $T < T_c$, Landau free energy shows two degenerate minima at $\eta = \pm n_s(T)$

Solving $\partial L/\partial\eta = 0$ for η we can read off the critical exponent β. We have :

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The critical exponent α of the heat capacity can be extracted by writing : $C_V = -T\partial^2 L/\partial T^2$, then :

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which shows that the heat capacity exhibit a discontinuity and that $\alpha = 0$

For the case $H > 0$. Taking the derivative with respect to H in Eq. 14 gives :

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where $\eta(H)$ is a solution of Eq. 15. For $t > 0$, we have $\eta = 0$, then $\chi_T \propto t^-1$ while for $t < 0$, we have $\eta = (-at/b)^{1/2}$ and $\chi_T \propto t^{-1}$. Thus, the critical exponent is $\gamma = 1$.

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The solution $\eta \neq 0$ is real when the argument of the square root is positive, i.e. $\sqrt{c^2 - at/b} > 0$. That is, $t < t^* = bc^2/a$

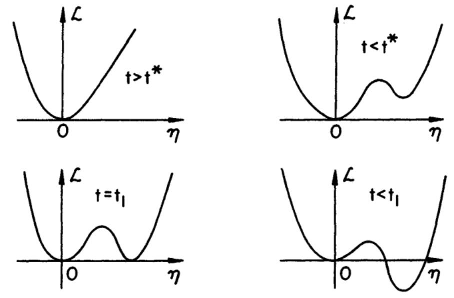
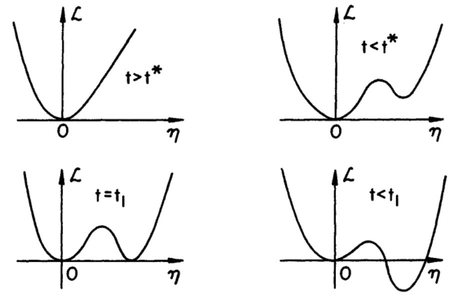


Figure 12: L as a function of η for different values of T showing how Landau's theory describes first order phase transitions



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A sufficient but not necessary condition of the occurrence of continuous phase transitions is that there are no cubic terms in the potential. In general, the cubic term causes a first order phase transition.

Thank you for your attendance and your attention.